http://www.doi.org/10.62341/istj-vol39-1-ja09

Received	**2026/06/26**	**تم استلام الورقة العلمية في**
Accepted	**2026/07/18**	**تم قبول الورقة العلمية في**
Published	**2026/07/19**	**تم نشر الورقة العلمية في**

Existence of Local Solutions to Coupled Chemotaxis-Fluid Equations with Chemotactic Sensitivity and Diffusion

Jameela Ebreed

Department of Mathematics,
Faculty of Science
University of Zawia, Libya
j.ebreed@zu.edu.ly

https://orcid.org/0009-0002-5028-1421

Aesha Lagha

Department of Mathematics,
Faculty of Science
University of Zawia, Libya
a.lagha@zu.edu.ly

https://orcid.org/0009-0002-5972-7389

Abstract

In this paper, we consider Keller-Segel chemotaxis coupled with the Navier-Stokes in three-dimensional. mechanism involving a diffusive partial differential equation. Which describes the dynamics of oxygen diffusion and consumption, chemotaxis, and viscous incompressible fluids. The cells and chemical substances are transported by a viscous incompressible fluid under the influence of a force due to the aggregation of cells. We use the energy method to prove the existence of local solutions in bounded domains of $\mathbb{R}^3$ for small initial.

Keywords: Chemotaxis system, Energy method, nonlinear diffusion.

Existence of Local Solutions to Coupled Chemotaxis- Fluid Equations with Chemotactic Sensitivity and Diffusion

http://www.doi.org/10.62341/istj-vol39-1-ja09

وجود الحلول المحلية لمعادلات الانجذاب الكيميائي–المائع المقترنة مع حساسية الانجذاب الكيميائي والانتشار

جميلة صالح عبريد

قسم الرياضيات، كلية العلوم، جامعة الزاوية، ليبيا.

عائشة المختار لاغة

قسم الرياضيات، كلية العلوم، جامعة الزاوية، ليبيا.

الملخص

في هذه الورقة، ندرس آلية كيلر–سيجل للانجذاب الكيميائي المقترنة بمعادلات نافيير–ستوكس في ثلاثة أبعاد، وهي آلية تتضمن معادلة تفاضلية جزئية انتشارية. تصف هذه المعادلة ديناميكيات انتشار الأكسجين واستهلاكه، والانجذاب الكيميائي، والسوائل اللزجة غير القابلة للانضغاط. تنتقل الخلايا والمواد الكيميائية بواسطة سائل لزج غير قابل للانضغاط تحت تأثير قوة ناتجة عن تجمع الخلايا. نستخدم طريقة الطاقة لإثبات وجود حلول محلية في نطاقات محدودة من الفضاء ثلاثي الأبعاد ($\mathbb{R}^3$) لقيم ابتدائية صغيرة.

الكلمات المفتاحية: نظام الانجذاب الكيميائي، طريقة الطاقة، الانتشار غير الخطي

1.1 Introduction.

In this paper, we consider the following couple system of the Keller-Segel type incompressible equations:

$$\begin{cases} \partial_t w + u.\nabla w = -\chi\nabla.(\mathrm{w}\nabla c) + D_w\Delta w \\ \partial_t c + wc + u.\nabla c = \ \varepsilon\Delta c \\ \partial_t u + u.\nabla u + \nabla p + \ w\nabla\varphi \ \ = D\Delta u \\ \nabla.u = 0 \qquad t > 0,\ x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

With initial data

http://www.doi.org/10.62341/istj-vol39-1-ja09

$$(w,c,u)|_{t=0} = \big(w_0(x), c_0(x), u_0(x)\big),\ \mathbb{R}^3, \tag{1.2}$$

where$\big(w_0(x), c_0(x), u_0(x)\big) \to (w_\infty, 0{,}0)$ as $|x| \to \infty$. Where $w = w(t,x), c = c(t,x), and\ u = u(t,x)$ denote the bacterial or calls density, the oxygen concentration, and the fluid velocity respectively. In addition, $p = p(t,x)$ is unknown pressure and φ is the gravitational potential function. The term $-\chi\nabla.(\mathrm{w}\nabla c)$ reflects the attractive movement of cells. While $w_0 = w_0(x)$, $c_0 = c_0(x)$, and $u_0 = u_0(x)$, are the given functions, the constants $\gamma > 0$, $\chi > 0$. Where n_∞ is a non-negative constant. A simple model case can be obtained upon the choices $\nabla\varphi = const.$, $\chi = 1$. The nonnegative constants D_w , ε and D are calls diffusion rates. The equation $\nabla.u = 0$, represents the condition of the incompressibility of the fluid, First two equations in system (1.1) describe the chemical movement of bacterial towards a higher concentration of oxygen to survive, where both bacteria and oxygen diffuse and are convicted with the fluid. The third equations in (1.1) are the Navier-Stokes equations, which describe the evolution of incompressible viscous fluids. With an additional term $w\nabla\varphi$ corresponding to the gravity force effect of bacterial calls on the fluid. Before going into further discussion of the chemotaxis fluid models, we would like to mention that the classical model for describing cell motion was proposed by Keller and Segel ([9], [10]). They derived a chemotaxis model that describes the directed movement of cells in response to specific chemicals in their environments. which play an essential role in various biological processes such as wound healing, cancer invasion, and predator avoidance. This model has received considerable attention due to its critical role in a wide range of biological phenomena. Their classical model describes this phenomenon through the following system of partial differential equations:

$$\begin{cases} u_t = k_1\Delta u - \nabla.(\mathrm{u}\chi(\mathrm{w})\nabla w) & x \in \Omega, t > 0, \\ w_t = k_2\Delta w - \alpha w + \beta u & x \in \Omega, t > 0, \end{cases} \tag{1.3}$$

http://www.doi.org/10.62341/istj-vol39-1-ja09

where $u(x,t) and\ w(x,t)$ represents the density of the bacteria and oxygen concentration, k_1, k_2 are the diffusion coefficients for cells and chemical respectively, χ quantifies the chemotactic sensitivity, while α and β represent the production and degradation rates of the chemoattractant. Analytic and numerical of Keller-Segel models with logistic source have been studied in the past few decades. The classical Keller-Segel model with logistic source of the form

$$\begin{cases} u_t = \Delta u - \chi\nabla.(\mathrm{u}\nabla v) + f(u) & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u & x \in \Omega, t > 0, \end{cases} \tag{1.4}$$

has been extensively studied in various contexts. Li X. [11] established global boundedness of classical solutions in whole space ($\mathbb{R}^{\mathrm{n}}$, $n \geq 1$) for nonnegative initial data, with particular emphasis on long-time behavior. Tuval et al. [16] conducted a detailed experimental and theoretical study on the interaction of bacterial chemotaxis, chemical diffusion, and fluid convection, the main responsible mechanisms are the chemical movement of bacteria towards the oxygen they consume and the effect of gravity on the movement of the fluid by heavier bacteria, and the thermal transport of both cells and oxygen through the fluid. In particular, by placing a water droplet containing Bacillus subtilis in a chamber with its upper surface open to the atmosphere, they observed that bacterial cells quickly get densely packed in a relatively thin liquid layer below the water-air interface through which the oxygen diffuses into the water droplet. For such processes, there is a mathematical system:

$$\begin{cases} n_t + u.\nabla n = \nabla.(D(n)\nabla n) - \nabla.(nS(x,n,c)\nabla c), & x \in \Omega, t > 0, \\ c_t + u.\nabla c = \Delta c - nf(c), & x \in \Omega, t > 0, \\ u_t + \kappa(u.\nabla u)u + \nabla p = \Delta u + n\nabla\varphi, & x \in \Omega, t > 0, \\ \nabla.u = 0\,, & x \in \Omega, t > 0, \end{cases} \tag{1.5}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary, where $f(c)$ is the rate of oxygen consumption, k measures the rate at which cells consume oxygen, and $S(x,n,c)$denotes a tensor-valued (or scalar) chemotactic sensitivity. Here n,u, p, φ, and $\kappa \in$ R

http://www.doi.org/10.62341/istj-vol39-1-ja09

denotes the bacteria density, the associated pressure of the fluid, the potential of the gravitational field, and the strength of nonlinear fluid convection, respectively. From the maximum principle of the parabolic equations, one can directly deduce that c is uniformly bounded from the second equation of (1.5). A lot of important progress has been made on the classical chemotaxis-Navier-Stokes equations, for which we refer to ([1],[2],[3]). Chunhua Jin [8] discussed the existence of global classical solution for the chemotaxis-Stokes system in a bounded domain in $\mathbb{R}^3$. Zheng [19] proved the system (1.6) has global existence of weak solutions. Winkler [17] showed that there exists at least one global weak solution in three-dimensional case. Lorz [13] shown the global existence of the solution for the small initial data and blow-up delay. Moreover, by constructing some proper free energy functional and deriving some uniform a priori estimates. Duan et al. [4] studied the global existence and decay rates of classical solutions over $\mathbb{R}^3$, provided that the initial datum (n_0, c_0, u_0) is a small smooth perturbation of the constant state $(n_\infty, 0, 0)$ with $n_\infty \geq 0$ in $H^3(\mathbb{R}^3)$. Hattori and Lagha ([6], [7]) showed the global existence and asymptotic behavior of the solutions for a chemotaxis system with chemoattractant and repellent in three dimensions. For more literature, one can refer to ([12],[14],[15],[18],[20]).

This paper is organized as follows, the first section is this introduction, which describes of some of the models used in chemotaxis, their rationale, and a very brief summary of the results obtained. In Section 2, we present notations and some assumptions that will be heavily used throughout the whole paper and state our main result. In Section 3, we prove the local-in-time existence of a regular solution for three-dimensional chemotaxis system with incompressible Navier-Stokes equations.

2. Preliminaries and Main result.

We first introduce some notations and useful inequalities that we will use later in this paper. We set $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ for a multi-index $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and length of α is $|.| = \alpha_3 + \alpha_2 + \alpha_3$. C and

http://www.doi.org/10.62341/istj-vol39-1-ja09

$\mathrm{C_i}$, where $i = 1,2$, denote some positive (generally small) constant, where both C and $\mathrm{C_i}$ may take different values in different places. For an integrable function $f: R^3 \to R$, let us denote the space

$$X(0,T) = \left\{ \begin{matrix} (w - w_\infty, c, u) \in \mathrm{C}\big([0,\mathrm{T}]; \mathrm{H}^3(\mathbb{R}^3)\big) \cap \mathrm{C}^1\big([0,\mathrm{T}]; \mathrm{H}^1(\mathbb{R}^3)\big), \\ \nabla(w - w_\infty), \nabla c, \nabla u \in \mathrm{L}^2\big([0,\mathrm{T}]; \mathrm{H}^3(\mathbb{R}^3)\big) \end{matrix} \right\}.$$

Definition 2.1 [5]

For $1 \le p < \infty$, the space $L^p(\Omega)$ consists of the Lebesgue measurable functions $f: \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f(x)|^p \, dx < \infty,$$

and $L^\infty(\Omega)$ consists of the essentially bounded function.
These spaces are Banach spaces with respect to the norms

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad \|f\|_\infty = \sup_{x\in\Omega} |f|$$

where sup denotes the essential supremum,

$$\sup_{x\in\Omega} |f| = \inf \{\mathrm{M} \ge 0 : \ |f| \le \mathrm{M} \ \text{ almost evrywhere in } \Omega\}.$$

Definition 2.2 [5]

The Sobolev space $W^{N,p}(U)$ consists of all summable functions $u: U \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Definition 2.3 [5].

If $u \in W^{N,p}(U)$, we define its norm to be

$$\|u\|_{W^{N,p}(U)} = \begin{cases} \left(\sum_{|\alpha|\le k} \int_\mathrm{U} |D^\alpha u|^p \, dx \right)^{\frac{1}{p}} & 1 \le p < \infty \\ \sum_{|\alpha|\le k} ess\ sup |D^\alpha u| & p = \infty. \end{cases}$$

http://www.doi.org/10.62341/istj-vol39-1-ja09

The Sobolev space $W^{N,2}(\mathbb{R}^3)$ is denoted as H^N. When $N = 0$, the norms in the space $L^2(\mathbb{R}^d)$ are denoted by $\|\cdot\|$. We will denote $\|\cdot\|_N$ is the H^N norm.

Lemma 2.1 (Cauchy's inequality with ϵ) [5]
Let a, b are any real numbers. Then

$$ab \leq \epsilon a^2 + \frac{b^2}{4\varepsilon} \qquad (a > 0, b > 0, \epsilon > 0).$$

Lemma 2.2 (Hölder's inequality) [5]

Assume $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. If $\mathrm{u} \in \mathrm{L}^{\mathrm{p}}(\Omega)$, then we have

$$\int_{\Omega} |\mathrm{uv}| \mathrm{dx}\,| \leq \|\mathrm{u}\|_{L^p(\Omega)} \|\mathrm{v}\|_{L^q(\Omega)}.$$

Lemma 2.3 (Gronwall's inequality differential) [5]
Let $f(t)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies the differential inequality
$f'(t) \leq g(t)f(t) + k(t)$, where $g(t)$ and $k(t)$, are nonnegative, summable functions on $[0, T]$. Then we have
$f(t) \leq e^{\int_0^t g(s)ds}[f(0) + \int_0^t k(s)ds]$ for all $0 \leq t \leq T$.

The main goal of this paper is to establish the existence of unique local solutions in three dimensions around a constant state $(w_\infty, 0, 0)$ for the above system (1.1). The main result of this paper is stated as follows:

Theorem 2.1. There exists a positive number ε_0 such that if

$$\|(w_0, c_0, u_0)\|_{\mathrm{H}^3} \leq \varepsilon_0,$$

http://www.doi.org/10.62341/istj-vol39-1-ja09

then the Cauchy problems (1.1)-(1.2) of the Keller-Segel system admits a unique local solution (w, c, u) with

$$(w - w_\infty, c, u) \in X(0, T).$$

The proof of the existence of local solutions in Theorem 2.2 by constructing a sequence of approximation functions based on iteration and some basic energy estimates. Let (w, c, u) be a smooth solution to the Cauchy problem of the Chemotaxis system (1.1) with initial data $U_0 = (w_0, c_0, u_0)$. We introduce the transformation:

$$w(t, x) = w_\infty + \sigma(t, x).$$

Then, the Cauchy problem (1.1) and (1.2) are reformulated as

$$\begin{cases} \partial_t \sigma + u.\nabla c + \nabla.(\sigma \nabla c) + w_\infty \nabla.(\nabla c) = D_w \Delta \sigma \\ \partial_t c + u.\nabla c + \sigma c + w_\infty c = \varepsilon \nabla c \\ \partial_t u + u.\nabla u + \nabla p + \sigma \nabla \varphi = D \Delta u \\ \nabla.u = 0 \quad t > 0, \qquad x \in \mathbb{R}^3, \end{cases} \tag{2.1}$$

with initial data

$$(\sigma, c, u)|_{t=0} = (\sigma_0, c_0, u_0) \to (0,0,0) \text{ as } |x| \to \infty, \tag{2.2}$$

where $\sigma_0 = w_0 - w_\infty$.

3. Existence of local solutions.

This section is devoted to the proof of Theorem 2.1. We construct the sequence $(\sigma^j, c^j, u^j)_{j \geq 0}$ by solving iteratively the Cauchy problems on the following linear equations

http://www.doi.org/10.62341/istj-vol39-1-ja09

$$\begin{cases} \partial_t \sigma^{j+1} + u^j.\nabla\sigma^{j+1} - D_w\Delta\sigma^{j+1} = -\nabla.\left(\sigma^j\nabla c^{j+1}\right) - w_\infty\Delta c^{j+1} \\ \partial_t c^{j+1} + u^j.\nabla c^{j+1} + w_\infty c^{j+1} - \varepsilon\Delta c^{j+1} = -\sigma^j c^{j+1} \\ \partial_t u^{j+1} + u^j.\nabla u^{j+1} + \nabla p^{j+1} - D\Delta u^{j+1} = -\sigma^j \nabla\varphi \\ \nabla.u^{j+1} = 0 \quad t > 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{3.1}$$

with initial data

$$\left(\sigma^{j+1}, c^{j+1}, u^{j+1}\right)|_{t=0} = (\sigma_0, c_0, u_0) \to (0,0,0),$$
(3.2)

as $|x| \to \infty$, for $j \geq 0$. In what follows, let us write $A^j = \left(\sigma^{j+1}, u^{j+1}, c^{j+1}\right)_{j\geq 0}$ and $A_0 = (\sigma_0, u_0, c_0)$, where $A^0 \equiv (0\,,0,0)$. Next, we prove that $(A^j)_{j\geq 0}$ is a Cauchy sequence in the Banach space $C([0,T_1];H^3)$ for $T_1 > 0$ suitable small. At last, by taking the limit and continuous argument, we prove that (σ, u, c) is a local solution to (2.1)-(2.2). Now, we will need to prove the following result:

Theorem 3.1.
Suppose $\|A_0\|_{H^3} \leq \varepsilon_1$, for small constants $\varepsilon_1 > 0$, $T_1 > 0$, $B_1 > 0$. Then for each $j \geq 0$, $A^j \in C([0, T_1)];\ H^3)$, is well defined and

$$\sup_{0\leq t\leq T_1} \left\|A^j(t)\right\|_{H^3} \leq B_1. \tag{3.3}$$

Moreover, $(A^j)_{j\geq 0}$ is a Cauchy sequence in Banach space $C([0, T_1]; H^3)$, the corresponding limit function denoted by A belongs to $C([0, T_1]; H^3)$ with

$$\sup_{0\leq t\leq T_1} \|A(t)\|_{H^3} \leq B_1, \tag{3.4}$$

http://www.doi.org/10.62341/istj-vol39-1-ja09

and A = (σ , u, c) is a solution to the Cauchy problem (2.1)-(2.2) over $[0, T_1]$. The Cauchy problem (2.1)-(2.2) admits at most one solution $A \in C([0,T_1]; H^3)$, which satisfies (3.4).

Proof. We begin by focusing our attention on the proof of (3.3), which will be given by an inductive argument. The trivial case is $j = 0$ since $A^0 = (0,0,0)$ by the assumption at initial step. Suppose that (3.3) holds for some $j \geq 0$ where is small enough. To prove (3.3) holds for j+1, we need some energy estimates on $(\sigma^{j+1}, c^{j+1}, u^{j+1})$. Apply ∂^α to both sides of the first equation of (3.1), multiply that equation by $\partial^\alpha \sigma^{j+1}$, and integrate over $\mathbb{R}^3$ with $|\alpha| \leq 3$, we obtain

$$\int_{\mathbb{R}^3} \partial^\alpha \sigma^{j+1} \partial^\alpha \partial_t \sigma^{j+1}\, dx + \int_{\mathbb{R}^3} \partial^\alpha\left(u^j . \nabla \sigma^{j+1}\right) \partial^\alpha \sigma^{j+1}\, dx$$
$$- D_w \int_{\mathbb{R}^3} \partial^\alpha \sigma^{j+1} \partial^\alpha\, \Delta \sigma^{j+1}\, dx$$
$$= - \int_{\mathbb{R}^3} \partial^\alpha \nabla . \left(\sigma^j \nabla c^{j+1}\right) \partial^\alpha \sigma^{j+1}\, dx$$
$$- w_\infty \int_{\mathbb{R}^3} \partial^\alpha \Delta c^{j+1} \partial^\alpha \sigma^{j+1} dx.$$

We use integration by parts, we get

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \left|\partial^\alpha \sigma^{j+1}\right|^2 dx + D_w \int_{\mathbb{R}^3} \left|\partial^\alpha \nabla \sigma^{j+1}\right|^2 dx$$
$$= - \int_{\mathbb{R}^3} \partial^\alpha\left(u^j . \nabla \sigma^{j+1}\right) \partial^\alpha \sigma^{j+1}\, dx$$
$$+ \int_{\mathbb{R}^3} \partial^\alpha\left(\sigma^j \nabla c^{j+1}\right) \partial^\alpha \nabla \sigma^{j+1}\, dx$$
$$+ w_\infty \int_{\mathbb{R}^3} \partial^\alpha \nabla \sigma^{j+1} \partial^\alpha \nabla c^{j+1} dx. \qquad (3.5)$$

http://www.doi.org/10.62341/istj-vol39-1-ja09

By taking the summation over $|\alpha| \leq 3$ and using Lemma 2.1, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\sigma^{j+1}\right\|_{H^3}^2 + C_1\left\|\nabla\sigma^{j+1}\right\|_{H^3}^2$$

$$\leq C\left\|\nabla c^{j+1}\right\|_{H^3}^2 + C\left\|(u^j,\sigma^j)\right\|_{H^3}^2\left\|\nabla\left(\sigma^{j+1},c^{j+1}\right)\right\|_{H^3}^2 + C\left\|\sigma^j\right\|_{H^3}^2. \qquad (3.6)$$

Similarly, applying ∂^α to the second equation of (3.1), multiplying it by $\partial^\alpha c^{j+1}$ and taking integrations in x, we obtain

$$\int_{\mathbb{R}^3}\partial^\alpha c^{j+1}\,\partial^\alpha\partial_t c^{j+1}\,dx + \int_{\mathbb{R}^3}\partial^\alpha c^{j+1}\,\partial^\alpha\left(u^j.\nabla c^{j+1}\right)dx$$

$$+w_\infty\int_{\mathbb{R}^3}\partial^\alpha c^{j+1}\,\partial^\alpha c^{j+1}\,dx - \varepsilon\int_{\mathbb{R}^3}\partial^\alpha c^{j+1}\,\partial^\alpha\left(\Delta c^{j+1}\right)dx$$

$$= -\int_{\mathbb{R}^3}\partial^\alpha c^{j+1}\,\partial^\alpha\left(\sigma^j\,c^{j+1}\right)dx.$$

After taking the summation over $|\alpha| \leq 3$ and using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|\partial^\alpha c^{j+1}\right|^2 dx + \varepsilon\int_{\mathbb{R}^3}\left|\nabla\partial^\alpha c^{j+1}\right|^2 dx$$

$$+w_\infty\int_{\mathbb{R}^3}\left|\partial^\alpha c^{j+1}\right|^2 dx$$

$$= -\int_{\mathbb{R}^3}\partial^\alpha\left(u^j.\nabla c^{j+1}\right)\partial^\alpha c^{j+1}dx - \int_{\mathbb{R}^3}\partial^\alpha\left(\sigma^j c^{j+1}\right)\partial^\alpha c^{j+1}dx\,.$$

After using Lemma 2.1, we have

http://www.doi.org/10.62341/istj-vol39-1-ja09

$$\frac{1}{2}\frac{d}{dt}\left\|c^{j+1}\right\|_{H^3}^2 + C_1\left\|\nabla c^{j+1}\right\|_{H^3}^2 + C_2\left\|c^{j+1}\right\|_{H^3}^2 \leq C\left\|(u^j,\sigma^j)\right\|_{H^3}^2\left\|\nabla c^{j+1}\right\|_{H^3}^2. \tag{3.7}$$

In a similar way as above, we can estimate u^{j+1} as

$$\int_{\mathbb{R}^3}\partial^\alpha u^{j+1}\,\partial^\alpha\left(\partial_t u^{j+1}\right)dx + \int_{\mathbb{R}^3}\partial^\alpha\left(\nabla p^{j+1}\right).\partial^\alpha u^{j+1}dx - D\int_{\mathbb{R}^3}\partial^\alpha\left(\Delta u^{j+1}\right).\partial^\alpha u^{j+1}dx$$

$$= -\int_{\mathbb{R}^3}\partial^\alpha\left(u^j.\nabla u^{j+1}\right).\partial^\alpha u^{j+1}dx - \int_{\mathbb{R}^3}\partial^\alpha\left(\sigma^j\nabla\varphi\right).\partial^\alpha u^{j+1}dx.$$

By using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|\partial^\alpha u^{j+1}\right|^2dx + D\int_{\mathbb{R}^3}\left|\nabla\,\partial^\alpha u^{j+1}\right|^2dx$$

$$= -\int_{\mathbb{R}^3}\partial^\alpha\left(u^j.\nabla u^{j+1}\right).\partial^\alpha u^{j+1}dx + \int_{\mathbb{R}^3}\partial^\alpha\left(\sigma^j\varphi\right).\partial^\alpha\nabla u^{j+1}dx.$$

Then, we take summation over $|\alpha| \leq 3$ and we use Lemma 2.1, one has

$$\frac{1}{2}\frac{d}{dt}\left\|u^{j+1}\right\|_{H^3}^2 + C_1\left\|\nabla u^{j+1}\right\|_{H^3}^2 \leq C\left\|\sigma^j\right\|_{H^3}^2 + C\left\|u^j\right\|_{H^3}^2\left\|\nabla u^{j+1}\right\|_{H^3}^2. \tag{3.8}$$

By combining (3.6), (3.7) and (3.8) we get

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\sigma^{j+1}\right\|_{H^3}^2 + \left\|c^{j+1}\right\|_{H^3}^2 + \left\|u^{j+1}\right\|_{H^3}^2\right)$$

http://www.doi.org/10.62341/istj-vol39-1-ja09

$$+C_1\left\|\nabla(\sigma^{j+1}, c^{j+1}, u^{j+1})\right\|_{H^3}^2 + C_2\left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2$$

$$\leq C\left\|\sigma^j\right\|_{H^3}^2 + C\left\|\sigma^j, c^j, u^j\right\|_{H^3}^2\left\|\nabla(\sigma^{j+1}, c^{j+1}, u^{j+1})\right\|_{H^3}^2. \quad (3.9)$$

Then, after integrating (3.9) over $[0, t]$ for all $t \in [0, T_1]$, we have

$$\begin{aligned}
&\left\|A^{j+1}(t)\right\|_{H^3}^2 + C_1\int_0^t \left\|\nabla A^{j+1}(s)\right\|_{H^3}^2 ds \\
&+ C_2\int_0^t \left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2 ds \\
&\leq C\left\|A_0\right\|_{H^3}^2 + C\int_0^t \left\|A^j(s)\right\|_{H^3}^2 ds \\
&+ C\int_0^t \left\|A^j(s)\right\|_{H^3}^2\left\|\nabla A^{j+1}(s)\right\|_{H^3}^2 ds, \quad (3.10)
\end{aligned}$$

for some positive constants C_1 *and* C_2. From the inductive assumption, the previous inequality can be re-estimated as

$$\left\|A^{j+1}(t)\right\|_{H^3}^2 + C_1\int_0^t \left\|\nabla A^{j+1}(s)\right\|_{H^3}^2 ds + C_2\int_0^t \left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2 ds$$
$$\leq C\varepsilon_1^2 + CB_1^2 T_1 + CB_1^2\int_0^t \left\|\nabla A^{j+1}(s)\right\|_{H^3}^2 ds, \quad (3.11)$$

for any $0 \leq t \leq T_1$. Now, we take the small constants $\varepsilon_1 > 0$, $B_1 > 0$ and $T_1 > 0$. Then, we get

$$\left\|A^{j+1}(t)\right\|_{H^3}^2 + C_1\left\|\nabla A^{j+1}(t)\right\|_{H^3}^2 + C_2\left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2 \leq B_1^2, \quad (3.12)$$

for any $0 \leq t \leq T_1$. This implies that (3.3) holds for $j+1$, hence (3.3) is proved for all $j \geq 0$.
Next, we define

http://www.doi.org/10.62341/istj-vol39-1-ja09

$$E(A^{j+1}(t)) := \|\sigma^{j+1}\|_{H^3}^2 + \|c^{j+1}\|_{H^3}^2 + \|u^{j+1}\|_{H^3}^2.$$

Similar to prove (3.10), we have

$$\left|E(A^{j+1}(t)) - E(A^{j+1}(s))\right| = \left|\int_s^t \frac{d}{d\tau} E(A^{j+1}(\tau))\, d\tau\right|$$
$$\leq C\int_s^t \|A^j(\tau)\|_{H^3}^2 d\tau +$$
$$C\int_s^t (1 + \|A^j(\tau)\|_{H^3}^2)\|\nabla A^{j+1}(\tau)\|_{H^3}^2 d\tau + C_2\int_0^t \|\sigma^{j+1}, c^{j+1}\|_{H^3}^2 d\tau \leq$$
$$CB_1{}^2(t-s) + C(1 + B_1{}^2)\int_s^t \|\nabla A^{j+1}(\tau)\|_{H^3}^2\, d\tau +$$
$$C_2\int_0^t \|\sigma^{j+1}, c^{j+1}\|_{H^3}^2 d\tau, \tag{3.13}$$

for any $0 \leq s \leq t \leq T_1$. Here, the time integral on the right-hand side from the above inequality is bounded by (3.12), and hence $E(A^{j+1}(t))$ is Continuous in t for each $j \geq 0$. By the same argument, we can infer that both $\|c^{j+1}\|_{H^3}^2$, and $\|u^{j+1}\|_{H^3}^2$ are continuous in t. From the continuity of $E\,(A^{j+1}(t))$, we can also infer the continuity of $\|\sigma^{j+1}\|_{H^3}^2$. Therefore,
$\|A^{j+1}(t)\|_{H^3}^2$ is continuous in time for $j \geq 1$.
For this step, we prove that the sequence $(A^j)_{j\geq 0}$ is a Cauchy sequence in the Banach space $C([0, T_1]; H^3)$, which converges to the solution $A = (\sigma, u, c)$ of the Cauchy problem (2.1)-(2.2), and satisfies (3.4). Let us take the difference of (3.1) for $j+1$ and j so that it gives

$$\begin{cases} \partial_t(\sigma^{j+1} - \sigma^j) - D_w\Delta(\sigma^{j+1} - \sigma^j) = -u^j.\nabla(\sigma^{j+1} - \sigma^j) - (u^j - u^{j-1}).\nabla\sigma^j \\ -\nabla.\left(\sigma^j\nabla(c^{j+1} - c^j)\right) - \nabla.((\sigma^j - \sigma^{j-1})\nabla c^j) \ - w_\infty(\Delta(c^{j+1} - c^j)) \\ \partial_t(c^{j+1} - c^j) - \varepsilon\Delta(c^{j+1} - c^j) + w_\infty(c^{j+1} - c^j) \\ = -u^j.\nabla(c^{j+1} - c^j) - (u^j - u^{j-1}).\nabla c^j - \sigma^j(c^{j+1} - c^j) - (\sigma^j - \sigma^{j-1})c^j, \\ \partial_t(u^{j+1} - u^j) - D\Delta(u^{j+1} - u^j) \ = -\nabla(p^{j+1} - p^j) \\ -u^j.\nabla(u^{j+1} - u^j) - (u^j - u^{j-1}).\nabla u^j - (\sigma^j - \sigma^{j-1})\nabla\varphi \\ \nabla.(u^{j+1} - u^j) = 0, \qquad t > 0, x \in \mathbb{R}^3. \end{cases} \tag{3.14}$$

http://www.doi.org/10.62341/istj-vol39-1-ja09

By using the same energy estimates as before, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|(\sigma^{j+1}-\sigma^{j})\right\|_{H^3}^{2}+C_1\left\|\nabla(\sigma^{j+1}-\sigma^{j})\right\|_{H^3}^{2}\\ \leq C\left\|\left(u^{j}-u^{j-1},\sigma^{j}-\sigma^{j-1}\right)\right\|_{H^3}^{2}\left\|\nabla(\sigma^{j},c^{j})\right\|_{H^3}^{2}\\ +C\left\|(\sigma^{j},u^{j})\right\|_{H^3}^{2}\left\|\nabla(\sigma^{j+1}-\sigma^{j},c^{j+1}\right.\\ \left.-c^{j})\right\|_{H^3}^{2}+w_{\infty}\left\|\nabla(c^{j+1}-c^{j})\right\|_{H^3}^{2}, \qquad (3.15)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|(c^{j+1}-c^{j})\right\|_{H^3}^{2}+C_2\left\|(c^{j+1}-c^{j})\right\|_{H^3}^{2}\\ +C_1\left\|\nabla(c^{j+1}-c^{j})\right\|_{H^3}^{2}\leq C\left\|(\sigma^{j},u^{j})\right\|_{H^3}^{2}\left\|\nabla(c^{j+1}-c^{j})\right\|_{H^3}^{2}\\ +C\left\|\nabla(\sigma^{j}-\sigma^{j-1},u^{j}-u^{j-1})\right\|_{H^3}^{2}\left\|\nabla c^{j}\right\|_{H^3}^{2}, \qquad (3.16)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|(u^{j+1}-u^{j})\right\|_{H^3}^{2}+C_1\left\|\nabla(u^{j+1}-u^{j})\right\|_{H^3}^{2}\\ \leq C\left\|u^{j}\right\|_{H^3}^{2}\left\|\nabla(u^{j+1}-u^{j})\right\|_{H^3}^{2}+C\left\|(u^{j}-u^{j-1})\right\|_{H^3}^{2}\left\|\nabla u^{j}\right\|_{H^3}^{2}\\ +C\left\|\sigma^{j}-\sigma^{j-1}\right\|_{H^3}^{2}. \qquad (3.17)\end{aligned}$$

We combine the equations (3.15),(3.16) and (3.17) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E\left(A^{j+1}-A^{j}\right)+C_1\left\|\nabla(A^{j+1}-A^{j})\right\|_{H^3}^{2}\\ +C_2\left\|(c^{j+1}-c^{j})\right\|_{H^3}^{2}+C_2\left\|(\sigma^{j}-\sigma^{j-1})\right\|_{H^3}^{2}\\ \leq C\left\|A^{j}\right\|_{H^3}^{2}\left\|\nabla(A^{j+1}-A^{j})\right\|_{H^3}^{2}\\ +C\left\|A^{j}\right.\\ \left.-A^{j-1}\right\|_{H^3}^{2}\left\|\nabla A^{j}\right\|_{H^3}^{2}. \qquad (3.18)\end{aligned}$$

By integrating over $[0,t]$ for any $0\leq t\leq T_1$ from (3.18), we have

http://www.doi.org/10.62341/istj-vol39-1-ja09

$$
\begin{aligned}
&\left\|(A^{j+1}(t)-A^{j}(t)\right\|_{H^3}^{2}+C_1\int_0^t\left\|\nabla(\mathrm{A}^{j+1}(s)-\mathrm{A}^{j}(s))\right\|_{\mathrm{H}^3}^{2}ds\\
&\quad+C_2\int_0^t\left\|(\mathrm{c}^{j+1}-\mathrm{c}^{j})\right\|_{\mathrm{H}^3}^{2}ds\\
&\quad+C_2\int_0^t\left\|(\sigma^{j+1}-\sigma^{j})\right\|_{\mathrm{H}^3}^{2}ds\\
&\quad\le\ C\left(1+{B_1}^2\right)T_1\sup_{0\le t\le T_1}\left\|(A^{j}(t)-A^{j-1}(t)\right\|_{H^3}\\
&\quad+C{B_1}^2\int_0^t\left\|\nabla(\mathrm{A}^{j+1}(s)-\mathrm{A}^{j}(s))\right\|_{\mathrm{H}^3}^{2}ds,
\end{aligned}
$$

which by smallness of B_1 and T_1 implies that there exists a constant $\theta\epsilon(0,1)$, such that for any $j\ge 1$

$$
\sup_{0\le t\le T_1}\left\|(A^{j+1}(t)-A^{j}(t)\right\|_{H^3}\le\theta\sup_{0\le t\le T_1}\left\|(A^{j}(t)-A^{j-1}(t)\right\|_{H^3}.
$$

It can be seen from the above inequality that $(\mathrm{A}^{j})_{j\ge 0}$ is a Cauchy sequence in the Banach space $C\left([0,T_1];H^3(\mathbb{R}^3)\right)$. Thus the limit function

$$
\mathrm{A}=\mathrm{A}^0+\lim_{i\to\infty}\sum_{j=0}^{i}\left(\mathrm{A}^{j+1}-\mathrm{A}^{j}\right)
$$

exists in $C\left([0,T_1];H^3(\mathbb{R}^3)\right)$, and satisfies

$$
\sup_{0\le t\le T_1}\|A(t)\|_{H^3}\le\sup_{0\le t\le T_1}\lim_{j\to\infty}inf\left\|A^{j}(t)\right\|_{H^3}\le B_1. \tag{3.19}
$$

Finally, we show that the Cauchy problem (2.1)-(2.2) admits at most one solution in $C([0,T_1];H^3(\mathbb{R}^3))$. Suppose that there exist two solutions $A,\tilde{A}$ in $C([0,T_1];H^3(\mathbb{R}^3))$ which satisfy (3.4). Let $\tilde{\sigma}=\sigma_1(x,t)-\sigma_2(x,t),\tilde{u}=u_1(x,t)-u_2(x,t),$ and $\tilde{c}=c_1(x,t)-c_2(x,t)$ solves

http://www.doi.org/10.62341/istj-vol39-1-ja09

$$\begin{cases} \partial_t\tilde{\sigma} + w_\infty\nabla.(\nabla\tilde{c}) - D_w\Delta\tilde{\sigma} = -u_1.\nabla\tilde{c} - \tilde{u}.\nabla c_2 - \nabla.(\sigma_2\,\nabla\tilde{c}) - \nabla.(\tilde{\sigma}\nabla c_1) \\ \partial_t\tilde{u} - D\Delta\tilde{u} - \nabla\tilde{p} = -u_2\nabla\cdot\tilde{u} - \tilde{u}\nabla\cdot u_1 - \tilde{\sigma}\nabla\varphi \\ \partial_t\tilde{c} - \varepsilon\Delta\tilde{c} + w_\infty\tilde{c} = -u_1\nabla\cdot\tilde{c} - \tilde{u}.\nabla c_2 - \tilde{\sigma}c_2 - \sigma_1\tilde{c}. \end{cases} \tag{3.20}$$

Multiplying $\tilde{\sigma}$ to both sides of the first equation of (3.20) and integrating over $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3}\tilde{\sigma}\,\partial_t\tilde{\sigma}dx + w_\infty\int_{\mathbb{R}^3}\tilde{\sigma}\,\nabla.(\nabla\tilde{c})\,dx - D_w\int_{\mathbb{R}^3}\tilde{\sigma}\,\Delta\tilde{\sigma}dx$$

$$= -\int_{\mathbb{R}^3}\tilde{\sigma}\,\nabla.(\tilde{\sigma}\nabla c_1)dx - \int_{\mathbb{R}^3}\tilde{\sigma}\,\nabla.(\sigma_2\nabla\tilde{c})dx$$

$$-\int_{\mathbb{R}^3}\tilde{\sigma}\,(u_1.\nabla\tilde{c}\,)dx - \int_{\mathbb{R}^3}\tilde{\sigma}\,(\tilde{u}.\nabla c_2)dx.$$

Then, after using integration by parts and Lemma 2.1, we obtain

$$\begin{aligned} &\frac{d}{2\,dt}\|\tilde{\sigma}\|_{L^2}^2 + w_\infty\|\tilde{\sigma}\|_{L^2}^2 + D_w\|\nabla\tilde{\sigma}\|_{L^2}^2 \\ &\le C\|\nabla c_1\|_{L^\infty}\int_{\mathbb{R}^3}(|\tilde{\sigma}|^2 + |\nabla\tilde{\sigma}|^2)dx \\ &+ C\|\sigma_2\|_{L^\infty}\int_{\mathbb{R}^3}(\,|\nabla\tilde{\sigma}|^2 + |\nabla\tilde{c}|^2)dx \\ &+ C\|u_1\|_{L^\infty}\int_{\mathbb{R}^3}(|\tilde{\sigma}|^2 + |\nabla\tilde{c}|^2)dx \\ &+ C\|\nabla c_2\|_{L^\infty}\int_{\mathbb{R}^3}(\,|\tilde{\sigma}|^2 + |\tilde{u}|^2)dx\,. \end{aligned} \tag{3.21}$$

Multiplying $\tilde{u}$ to both sides of the second equation of (2.1) and taking integrations in x, we obtain

$$\int_{\mathbb{R}^3}\tilde{u}\,\partial_t\tilde{u}dx - \int_{\mathbb{R}^3}\tilde{u}\,\Delta\tilde{u}dx - \int_{\mathbb{R}^3}\tilde{u}\,\nabla\tilde{p}dx =$$

$$-\int_{\mathbb{R}^3}\tilde{u}\,.(\,\tilde{u}\nabla\cdot u_1)dx - \int_{\mathbb{R}^3}\tilde{u}\,.(u_2\nabla\cdot\tilde{u})dx - \int_{\mathbb{R}^3}\tilde{u}\,.(\tilde{\sigma}\nabla\varphi)dx.$$

http://www.doi.org/10.62341/istj-vol39-1-ja09

By using integration by parts and Lemma 2.1, we get

$$\frac{d}{2\,dt}\|\tilde{u}\|_{L^2}^2 + D\|\nabla\tilde{u}\|_{L^2}^2 \leq \mathrm{C}\|\nabla\cdot u_1\|_{L^\infty}\|\tilde{u}\|_{L^2}^2$$

$$+\mathrm{C}\|u_2\|_{L^\infty}(\|\tilde{u}\|_{L^2}^2+\|\nabla.\tilde{u}\|_{L^2}^2) + \mathrm{C}\|\nabla\varphi\|_{L^\infty}\left(\|\tilde{u}\|_{L^2}^2 + \|\tilde{\sigma}\|_{L^2}^2\right).$$

Since L^∞ norms of σ^i,u^i, and c^i where i = 1, 2 are bounded, we have

$$\frac{d}{2\,dt}\|\tilde{u}\|_{L^2}^2 + \mathrm{C}_1\|\nabla\tilde{u}\|_{L^2}^2 \leq \mathrm{C}\|\tilde{\sigma}\|_{L^2}^2 + \mathrm{C}\|\tilde{u}\|_{L^2}^2. \tag{3.22}$$

Similarly, we estimate $\tilde{c}$ as follows:

$$\frac{d}{2\,dt}\|\tilde{c}\|_{L^2}^2 + \mathrm{C}_1\|\nabla\tilde{c}\|_{L^2}^2 + \mathrm{C}_2\|\tilde{c}\|_{L^2}^2 \leq \mathrm{C}\left(\|\tilde{c}\|_{L^2}^2 + \|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2\right). \tag{3.23}$$

After taking the linear combination of (3.21),(3.22), and (3.23), we obtain

$$\frac{d}{2\,dt}\left(\|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2\right) + \mathrm{C}_2\left(\|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2\right) + \mathrm{C}_1\left(\|\nabla\tilde{\sigma}\|_{L^2}^2+\|\nabla\tilde{u}\|_{L^2}^2 + \|\nabla\tilde{c}\|_{L^2}^2\right) \leq C\left(\|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2\right). \tag{3.24}$$

By applying Lemma 2.3, to the above equation, we have

$$\sup_{0\leq t\leq T_1}\left(\|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2\right) \leq e^{cT_1}\left(\|\tilde{\sigma}(0)\|_{L^2}^2 + \|\tilde{u}(0)\|_{L^2}^2 + \|\tilde{c}(0)\|_{L^2}^2\right).$$

Since the initial data of ($\tilde{\sigma}$, $\tilde{u}$,$\tilde{c}$) are all zero for T > 0, that implies the uniqueness of the local solution.

http://www.doi.org/10.62341/istj-vol39-1-ja09

Conclusion

In this paper, we proved the existence of local solutions for chemotaxis- fluid equations with chemotactic sensitivity and diffusion, which described modeling cellular swimming in fluid drops in three dimensions. We obtained the local existence and uniqueness of convergence on classical solutions near constant states by the energy method.

References

[1] Chae, M., Kang K., Lee J. (2012), Existence of smooth solutions to coupled chemotaxis-
fluid equations, Discrete and continuous dynamical systems, 33, 2271-2297.

[2] Chae, M., Kang K., Lee. J. (2014), Global existence and temporal decay in Keller-Segel models coupled to fluid equations, Comm. Partial Di. Equations, 39, 1205-1235.

[3] DiFrancesco, M., Lorz, A., Markowich P.A. (2010), Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior. Discr. Cont. Syst. A28, 1437-1453.

[4] Duan R., Lorz, A., Markowich P. (2010), Global solutions to the coupled chemotaxis-fluid equations, Comm. Partial Diff. Equations, 35, 1635-1673.

[5] Evans L. C. (1998), *Partial Differential Equations,* American Mathematical Society ISBN 9780821848593.

[6] Hattori H., Lagha A. (2021), Existence of global solutions to chemotaxis fluid system with logistic source. Electronic Journal of Qualitative Theory of Differential Equations, vol. 2022, No. 01-87.

[7] Hattori H., Lagha, A. (2021), Global Existence and Decay Rates of the Solutions for a Chemotaxis System Lotka-Volterra Type Model for Chemo Agents. Discrete and continuous dynamical systems,Vol.41, 2021.

http://www.doi.org/10.62341/istj-vol39-1-ja09

[8] Jin Ch., (2018), Global classical solution and stability to a coupled chemotaxis-fluid model with logistic source. Discrete and continuous dynamical systems, 38(7): 3547-3566.

[9] Keller E. F. and L. A. Segel (1971), A model for chemotaxis, J. Theor. Biol., 30, 225-234.

[10] Keller E. F. and L. A. Segel (1970), Initiation of slide mold aggregation viewed as an instability, J. Theor. Biol., 26, 399-415. with logistic source, Discrete and continuous dynamical systems, V38, 3547-3566.

[11] Li X. (2019), On a fully parabolic chemotaxis system with nonlinear signal secretion, Nonlinear Anal. Real World Appl., 49, 24-44.

[12] Liu D., Tao Y.(2016), Boundedness in a chemotaxis system with nonlinear signal production, Appl. Math. J. Chinese Univ. Ser. B, 31, 379-388.

[13] Lorz A. (2012), A coupled Keller-Segel-Stokes model: global existence for small initial data and blow-up delay. Commun Math Sci., 10, 555-574.

[14] Osaki K., Yagi A.(2001), Finite dimensional attractor for one-dimensional Keller-Segel equations, Funkcial. Ekvac., 44, 441-469.

[15] Shi S., Liu Z. and Jin H. (2017), Boundedness and large time behavior of an attraction-repulsion chemotaxis model with logistic source, Kinet. Relat. Models, 10, 855-878.

[16] Tuval, I., Cisneros, L., Dombrowski, C., Wolgemuth, C. W., Kessler. J. O., Goldstein, R. E. (2005), Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. USA, 102, 2277-2282.

[17] Winkler, M. (2014), Stabilization in a two-dimensional chemotaxis-Navier-Stokes system, Arch. Rational Mech. Anal., 211, 455-487.

[18] Winkler, M. (2012), Global large-data solutions in a chemotaxis-(Navier-Stokes) system modeling cellular

http://www.doi.org/10.62341/istj-vol39-1-ja09

swimming in fluid drops, Comm. Partial Differential Equations, 37, 319-35.
[19] Zhang Q., Li Y. (2015), Convergence rates of solutions for a two-dimensional chemotaxis-Navier-Stokes system, Discrete Contin. Dyn. Syst. B, 20, 2751-2759.
[20] Zhang Q., Zhang Y. (2019), On the global well-posedness for the 2D incompressible Keller-Segel-Navier-Stokes equations, Z. Angew. Math. Mech., 11, 99.